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Non-properly embedded H-planes in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract For any $H \in (0, \frac{1}{2})$, we construct complete, non-proper, stable, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature H.

1 Introduction

In their ground breaking work [2], Colding and Minicozzi proved that complete minimal surfaces embedded in $\mathbb{R}^3$ with finite topology are proper. Based on the techniques in [2], Meeks and Rosenberg [5] then proved that complete minimal surfaces with positive injectivity embedded in $\mathbb{R}^3$ are proper. More recently, Meeks and Tinaglia [7]

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proved that complete constant mean curvature surfaces embedded in $\mathbb{R}^3$ are proper if they have finite topology or have positive injectivity radius.

In contrast to the above results, in this paper we prove the following existence theorem for non-proper, complete, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in (0, 1/2)$. The convention used here is that the mean curvature function of an oriented surface M in an oriented Riemannian three-manifold N is the pointwise average of its principal curvatures.

The catenoids in $\mathbb{H}^2 \times \mathbb{R}$ mentioned in the next theorem are defined at the beginning of Sect. 2.1.

Theorem 1.1 *For any $H \in (0, 1/2)$ there exists a complete, stable, simply-connected surface Σ_H embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature H satisfying the following properties:*

(1) The closure of Σ_H is a lamination with three leaves, Σ_H, C_1 and C_2, where C_1 and C_2 are stable catenoids of constant mean curvature H in $\mathbb{H}^3$ with the same axis of revolution L. In particular, Σ_H is not properly embedded in $\mathbb{H}^2 \times \mathbb{R}$.

(2) Let K_L denote the Killing field generated by rotations around L. Every integral curve of K_L that lies in the region between C_1 and C_2 intersects Σ_H transversely in a single point. In particular, the closed region between C_1 and C_2 is foliated by surfaces of constant mean curvature H, where the leaves are C_1 and C_2 and the rotated images $\Sigma_H(\theta)$ of Σ around L by angle $\theta \in [0, 2\pi)$.

When $H = 0$, Rodríguez and Tinaglia [10] constructed non-proper, complete minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$. However, their construction does not generalize to produce complete, non-proper planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ with non-zero constant mean curvature. Instead, the construction presented in this paper is related to the techniques developed by the authors in [3] to obtain examples of non-proper, stable, complete planes embedded in $\mathbb{H}^3$ with constant mean curvature H, for any $H \in [0, 1)$.

There is a general conjecture related to Theorem 1.1 and the previously stated positive properness results. Given X a Riemannian three-manifold, let $\mathrm{Ch}(X) := \inf_{S \in \mathcal{S}} \frac{\mathrm{Area}(\partial S)}{\mathrm{Volume}(S)}$, where $\mathcal{S}$ is the set of all smooth compact domains in X. Note that when the volume of X is infinite, $\mathrm{Ch}(X)$ is the Cheeger constant.

Conjecture 1.2 *Let X be a simply-connected, homogeneous three-manifold. Then for any $H \geq \frac{1}{2}Ch(X)$, every complete, connected H-surface embedded in X with positive injectivity radius or finite topology is proper. On the other hand, if $Ch(X) > 0$, then there exist non-proper complete H-planes in X for every $H \in [0, \frac{1}{2}Ch(X))$.*

By the work in [2], Conjecture 1.2 holds for $X = \mathbb{R}^3$ and it holds in $\mathbb{H}^3$ by work in progress in [6]. Since the Cheeger constant of $\mathbb{H}^2 \times \mathbb{R}$ is 1, Conjecture 1.2 would imply that Theorem 1.1 (together with the existence of complete non-proper minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ found in [10]) is a sharp result.

2 Preliminaries

In this section, we will review the basic properties of H-surfaces, a concept that we next define. We will call a smooth oriented surface Σ_H in $\mathbb{H}^2 \times \mathbb{R}$ an *H-surface* if it is embedded and its mean curvature is constant equal to H; we will assume that Σ_H is appropriately oriented so that H is non-negative. We will use the cylinder model of $\mathbb{H}^2 \times \mathbb{R}$ with coordinates (ρ, θ, t); here ρ is the hyperbolic distance from the origin (a chosen base point) in $\mathbb{H}^2_0$, where $\mathbb{H}^2_t$ denotes $\mathbb{H}^2 \times \{t\}$. We next describe the H-catenoids mentioned in the Introduction.

The following H-catenoids family will play a particularly important role in our construction.

2.1 Rotationally invariant vertical H-catenoids $\mathcal{C}_d^H$

We begin this section by recalling several results in [8,9]. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, let

$$\eta_d = \cosh^{-1}\left(\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}\right)$$

and let $\lambda_d \colon [\eta_d, \infty) \to [0, \infty)$ be the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr. \tag{1}$$

Note that $\lambda_d(\rho)$ is a strictly increasing function with $\lim_{\rho\to\infty} \lambda_d(\rho) = \infty$ and derivative $\lambda_d'(\eta_d) = \infty$ when $d \in (-2H, \infty)$.

In [8] Nelli, Sa Earp, Santos and Toubiana proved that there exists a 1-parameter family of embedded H-catenoids $\{\mathcal{C}_d^H \mid d \in (-2H, \infty)\}$ obtained by rotating a generating curve $\lambda_d(\rho)$ about the t-axis. The generating curve $\widehat{\lambda}_d$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$. Note that $\widehat{\lambda}_d$ is a smooth curve and that the necksize, η_d, is a strictly increasing function in d satisfying the properties that $\eta_{-2H} = 0$ and $\lim_{d\to\infty} \eta_d = \infty$.

If $d = -2H$, then by rotating the curve $(\rho, 0, \lambda_d(\rho))$ around the t-axis one obtains a simply-connected H-surface E_H that is an entire graph over $\mathbb{H}^2_0$. We denote by $-E_H$ the reflection of E_H across $\mathbb{H}^2_0$.

We next recall the definition of the mean curvature vector.

Definition 2.1 Let M be an oriented surface in an oriented Riemannian three-manifold and suppose that M has non-zero mean curvature $H(p)$ at p. The **mean curvature vector at** p is $\mathbf{H}(p) := H(p)N(p)$, where $N(p)$ is its unit normal vector at p. The mean curvature vector $\mathbf{H}(p)$ is independent of the orientation on M.

Note that the mean curvature vector $\mathbf{H}$ of $\mathcal{C}_d^H$ points into the connected component of $\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}_d^H$ that contains the t-axis. The mean curvature vector of E_H points upward while the mean curvature vector of $-E_H$ points downward.

In order to construct the examples described in Theorem 1.1, we first obtain certain geometric properties satisfied by H-catenoids. For example, in the following lemma, we show that for certain values of d_1 and d_2, the catenoids $\mathcal{C}_{d_1}^H$ and $\mathcal{C}_{d_2}^H$ are disjoint.

Given $d \in (-2H, \infty)$, let $b_d(t) := \lambda_d^{-1}(t)$ for $t \geq 0$; note that $b_d(0) = \eta_d$. Abusing the notation let $b_d(t) := b_d(-t)$ for $t \leq 0$.

Lemma 2.1 *(Disjoint H-catenoids) Given $d_1 > 2$, there exist $d_0 > d_1$ and $\delta_0 > 0$ such that for any $d_2 \in [d_0, \infty)$, then*

$$\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.$$

In particular, the corresponding H-catenoids are disjoint, i.e. $\mathcal{C}_{d_1}^H \cap \mathcal{C}_{d_2}^H = \emptyset$.

Moreover, $b_{d_2}(t) - b_{d_1}(t)$ is decreasing for $t > 0$ and increasing for $t < 0$. In particular,

$$\sup_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.$$

The proof of the above lemma requires a rather lengthy computation that is given in the Appendix.

We next recall the well-known mean curvature comparison principle.

Proposition 2.2 *(Mean curvature comparison principle) Let M_1 and M_2 be two complete, connected embedded surfaces in a three-dimensional Riemannian manifold. Suppose that $p \in M_1 \cap M_2$ satisfies that a neighborhood of p in M_1 locally lies on the side of a neighborhood of p in M_2 into which $\mathbf{H}_2(p)$ is pointing. Then $|H_1|(p) \geq |H_2|(p)$. Furthermore, if M_1 and M_2 are constant mean curvature surfaces with $|H_1| = |H_2|$, then $M_1 = M_2$.*

3 The examples

For a fixed $H \in (0, 1/2)$, the outline of construction is as follows. First, we will take two disjoint H-catenoids $\mathcal{C}_1$ and $\mathcal{C}_2$ whose existence is given in Lemma 2.1. These catenoids $\mathcal{C}_1$, $\mathcal{C}_2$ bound a region Ω in $\mathbb{H}^2 \times \mathbb{R}$ with fundamental group $\mathbb{Z}$. In the universal cover $\widetilde{\Omega}$ of Ω, we define a piecewise smooth compact exhaustion $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_n \subset \cdots$ of $\widetilde{\Omega}$. Then, by solving the H-Plateau problem for special curves $\Gamma_n \subset \partial \Delta_n$, we obtain minimizing H-surfaces Σ_n in Δ_n with $\partial \Sigma_n = \Gamma_n$. In the limit set of these surfaces, we find an H-plane $\mathcal{P}$ whose projection to Ω is the desired non-proper H-plane $\Sigma_H \subset \mathbb{H}^2 \times \mathbb{R}$.

3.1 Construction of $\widetilde{\Omega}$

Fix $H \in (0, \frac{1}{2})$ and $d_1, d_2 \in (2, \infty)$, $d_1 < d_2$, such that by Lemma 2.1, the related H-catenoids $\mathcal{C}_{d_1}^H$ and $\mathcal{C}_{d_2}^H$ are disjoint; note that in this case, $\mathcal{C}_{d_1}^H$ lies in the interior of the simply-connected component of $\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}_{d_2}^H$. We will use the notation $\mathcal{C}_i := \mathcal{C}_{d_i}^H$. Recall that both catenoids have the same rotational axis, namely the t-axis, and recall that the mean curvature vector $\mathbf{H}_i$ of $\mathcal{C}_i$ points into the connected component of

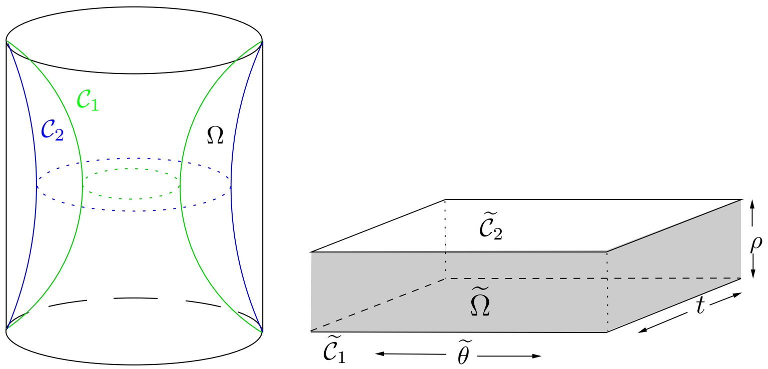

Fig. 1 The induced coordinates $(\rho, \widetilde{\theta}, t)$ in $\widetilde{\Omega}$

$\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}_i$ that contains the t-axis. We emphasize here that H is *fixed* and so we will omit describing it in future notations.

Let Ω be the closed region in $\mathbb{H}^2 \times \mathbb{R}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $\partial\Omega = \mathcal{C}_1 \cup \mathcal{C}_2$ (Fig. 1-left). Notice that the set of boundary points at infinity $\partial_\infty\Omega$ is equal to $S^1_\infty \times \{-\infty\} \cup S^1_\infty \times \{\infty\}$, i.e., the corner circles in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ in the product compactification, where we view $\mathbb{H}^2$ to be the open unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with base point the origin $\vec{0}$.

By construction, Ω is topologically a solid torus. Let $\widetilde{\Omega}$ be the universal cover of Ω. Then, $\partial\widetilde{\Omega} = \widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2$ (Fig. 1-right), where $\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{C}}_2$ are the respective lifts to $\widetilde{\Omega}$ of $\mathcal{C}_1, \mathcal{C}_2$. Notice that $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ are both H-planes, and the mean curvature vector $\mathbf{H}$ points outside of $\widetilde{\Omega}$ along $\widetilde{\mathcal{C}}_1$ while $\mathbf{H}$ points inside of $\widetilde{\Omega}$ along $\widetilde{\mathcal{C}}_2$. We will use the induced coordinates $(\rho, \widetilde{\theta}, t)$ on $\widetilde{\Omega}$ where $\widetilde{\theta} \in (-\infty, \infty)$. In particular, if

$$\Pi : \widetilde{\Omega} \to \Omega \tag{2}$$

is the covering map, then $\Pi(\rho_o, \widetilde{\theta}_o, t_o) = (\rho_o, \theta_o, t_o)$ where $\theta_o \equiv \widetilde{\theta}_o \mod 2\pi$.

Recalling the definition of $b_i(t)$, $i = 1, 2$, note that a point (ρ, θ, t) belongs to Ω if and only if $\rho \in [b_1(t), b_2(t)]$ and we can write

$$\widetilde{\Omega} = \{(\rho, \widetilde{\theta}, t) \mid \rho \in [b_1(t), b_2(t)],\ \widetilde{\theta} \in \mathbb{R},\ t \in \mathbb{R}\}.$$

3.2 Infinite bumps in $\widetilde{\Omega}$

Let γ be the geodesic through the origin in $\mathbb{H}^2_0$ obtained by intersecting $\mathbb{H}^2_0$ with the vertical plane $\{\theta = 0\} \cup \{\theta = \pi\}$. For $s \in [0, \infty)$, let φ_s be the orientation preserving hyperbolic isometry of $\mathbb{H}^2_0$ that is the hyperbolic translation along the geodesic γ with $\varphi_s(0, 0) = (s, 0)$. Let

$$\widehat{\varphi}_s : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}, \quad \widehat{\varphi}_s(\rho, \theta, t) = (\varphi_s(\rho, \theta), t) \tag{3}$$

be the related extended isometry of $\mathbb{H}^2 \times \mathbb{R}$.

Let $\mathcal{C}_d$ be an embedded H-catenoid as defined in Sect. 2.1. Notice that the rotation axis of the H-catenoid $\widehat{\varphi}_{s_0}(\mathcal{C}_d)$ is the vertical line $\{(s_0, 0, t) \mid t \in \mathbb{R}\}$.

Let $\delta := \inf_{t\in\mathbb{R}}(b_2(t) - b_1(t))$, which gives an upper bound estimate for the asymptotic distance between the catenoids; recall that by our choices of $\mathcal{C}_1, \mathcal{C}_2$ given in Lemma 2.1, we have $\delta > 0$. Let $\delta_1 = \frac{1}{2}\min\{\delta, \eta_1\}$ and let $\delta_2 = \delta - \frac{\delta_1}{2}$. Let $\widehat{\mathcal{C}}_1 := \widehat{\varphi}_{\delta_1}(\mathcal{C}_1)$ and $\widehat{\mathcal{C}}_2 := \widehat{\varphi}_{-\delta_2}(\mathcal{C}_2)$. Note that $\delta_1 + \delta_2 > \delta$.

Claim 3.1 *The intersection $\Omega \cap \widehat{\mathcal{C}}_i$, $i = 1, 2$, is an infinite strip.*

Proof Given $t \in \mathbb{R}$, let $\mathbb{H}^2_t$ denote $\mathbb{H}^2 \times \{t\}$. Let $\tau^i_t := \mathcal{C}_i \cap \mathbb{H}^2_t$ and $\widehat{\tau}^i_t := \widehat{\mathcal{C}}_i \cap \mathbb{H}^2_t$. Note that for $i = 1, 2$, τ^i_t is a circle in $\mathbb{H}^2_t$ of radius $b_i(t)$ centered at $(0, 0, t)$ while $\widehat{\tau}^1_t$ is a circle in $\mathbb{H}^2_t$ of radius $b_1(t)$ centered at $p_{1,t} := (\delta_1, 0, t)$ and $\widehat{\tau}^2_t$ is a circle in $\mathbb{H}^2_t$ of radius $b_2(t)$ centered at $p_{2,t} := (-\delta_2, 0, t)$. We claim that for any $t \in \mathbb{R}$, the intersection $\widehat{\tau}^i_t \cap \Omega$ is an arc with end points in τ^i_t, $i = 1, 2$. This result would give that $\Omega \cap \widehat{\mathcal{C}}_i$ is an infinite strip. We next prove this claim.

Consider the case $i = 1$ first. Since $\delta_1 < \eta_1 \leq b_1(t)$, the center $p_{1,t}$ is inside the disk in $\mathbb{H}^2_t$ bounded by τ^1_t. Since the radii of τ^1_t and $\widehat{\tau}^1_t$ are both equal to $b_1(t)$, then the intersection $\tau^1_t \cap \widehat{\tau}^1_t$ is nonempty. It remains to show that $\widehat{\tau}^1_t \cap \tau^2_t = \emptyset$, namely that $b_1(t) + \delta_1 < b_2(t)$. This follows because

$$\delta_1 < \delta = \inf_{t\in\mathbb{R}}(b_2(t) - b_1(t)).$$

This argument shows that $\Omega \cap \widehat{\mathcal{C}}_1$ is an infinite strip.

Consider now the case $i = 2$. Since $\delta_2 < \delta < b_2(t)$, the center $p_{2,t}$ is inside the disk in $\mathbb{H}^2_t$ bounded by τ^2_t. Since the radii of τ^2_t and $\widehat{\tau}^2_t$ are both equal to $b_2(t)$, then the intersection $\tau^2_t \cap \widehat{\tau}^2_t$ is nonempty. It remains to show that $\tau^1_t \cap \widehat{\tau}^2_t = \emptyset$, namely that $b_2(t) - \delta_2 > b_1(t)$. This follows because

$$b_2(t) - b_1(t) \geq \inf_{t\in\mathbb{R}}(b_2(t) - b_1(t)) = \delta > \delta_2$$

This completes the proof that $\Omega \cap \widehat{\mathcal{C}}_2$ is an infinite strip and finishes the proof of the claim. □

Now, let $Y^+ := \Omega \cap \widehat{\mathcal{C}}_2$ and let $Y^- := \Omega \cap \widehat{\mathcal{C}}_1$. In light of Claim 3.1 and its proof, we know that $Y^+ \cap \mathcal{C}_1 = \emptyset$ and $Y^- \cap \mathcal{C}_2 = \emptyset$.

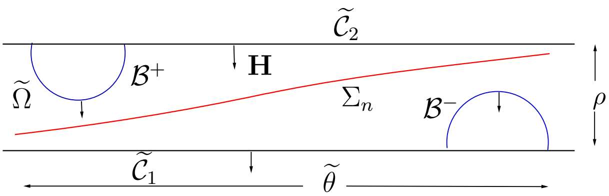

Fig. 2 The position of the bumps $\mathcal{B}^{\pm}$ in $\widetilde{\Omega}$ is shown in the picture. The *small arrows* show the mean curvature vector direction. The H-surfaces Σ_n are disjoint from the infinite strips $\mathcal{B}^{\pm}$ by construction

Remark 3.2 Note that by construction, any rotational surface contained in Ω must intersect $\widehat{\mathcal{C}}_1 \cup \widehat{\mathcal{C}}_2$. In particular, $Y^+ \cup Y^-$ intersects all H-catenoids $\mathcal{C}_d$ for $d \in (d_1, d_2)$ as the circles $\mathcal{C}_d \cap \mathbb{H}^2_t$ intersect either the circle $\widehat{\tau}^2_t$ or the circle $\widehat{\tau}^1_t$ for some $t > 0$ since $\delta_1 + \delta_2 > \delta$.

In $\widetilde{\Omega}$, let $\mathcal{B}^+$ be the lift of Y^+ in $\widetilde{\Omega}$ which intersects the slice $\{\widetilde{\theta} = -10\pi\}$. Similarly, let $\mathcal{B}^-$ be the lift of Y^- in $\widetilde{\Omega}$ which intersects the slice $\{\widetilde{\theta} = 10\pi\}$. Note that each lift of Y^+ or $Y^-$is contained in a region where the $\widetilde{\theta}$ values of their points lie in ranges of the form $(\theta_0 - \pi, \theta_0 + \pi)$ and so $\mathcal{B}^+ \cap \mathcal{B}^- = \emptyset$. See Fig. 2.

The H-surfaces $\mathcal{B}^\pm$ near the top and bottom of $\widetilde{\Omega}$ will act as barriers (infinite bumps) in the next section, ensuring that the limit H-plane of a certain sequence of compact H-surfaces does not collapse to an H-lamination of $\widetilde{\Omega}$ all of whose leaves are invariant under translations in the $\widetilde{\theta}$-direction.

Next we modify $\widetilde{\Omega}$ as follows. Consider the component of $\widetilde{\Omega} - (\mathcal{B}^+ \cup \mathcal{B}^-)$ containing the slice $\{\widetilde{\theta} = 0\}$. From now on we will call the **closure** of this region $\widetilde{\Omega}^*$.

3.3 The compact exhaustion of $\widetilde{\Omega}^*$

Consider the rotationally invariant H-planes $E_H, -E_H$ described in Sect. 2. Recall that E_H is a graph over the horizontal slice $\mathbb{H}^2_0$ and it is also tangent to $\mathbb{H}^2_0$ at the origin. Given $t \in \mathbb{R}$, let $E^t_H = -E_H + (0, 0, t)$ and $-E^t_H = E_H - (0, 0, t)$. Both families $\{E^t_H\}_{t \in \mathbb{R}}$ and $\{-E^t_H\}_{t \in \mathbb{R}}$ foliate $\mathbb{H}^2 \times \mathbb{R}$. Moreover, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $n \in \mathbb{N}$, the following holds. The highest (lowest) component of the intersection $S^+_n := E^n_H \cap \Omega$ ($S^-_n := -E^n_H \cap \Omega$) is a rotationally invariant annulus with boundary components contained in $\mathcal{C}_1$ and $\mathcal{C}_2$. The annulus S^+_n lies "above" S^-_n and their intersection is empty. The region $\mathcal{U}_n$ in Ω between S^+_n and S^-_n is a solid torus, see Fig. 3-left, and the mean curvature vectors of S^+_n and S^-_n point into $\mathcal{U}_n$.

Let $\widetilde{\mathcal{U}}_n \subset \widetilde{\Omega}$ be the universal cover of $\mathcal{U}_n$, see Fig. 3-right. Then, $\partial\widetilde{\mathcal{U}}_n - \partial\widetilde{\Omega} = \widetilde{S}^+_n \cup \widetilde{S}^-_n$ where can view $\widetilde{S}^\pm_n$ as a lift to $\widetilde{\mathcal{U}}_n$ of the universal cover of the annulus $S^\pm_n$. Hence,

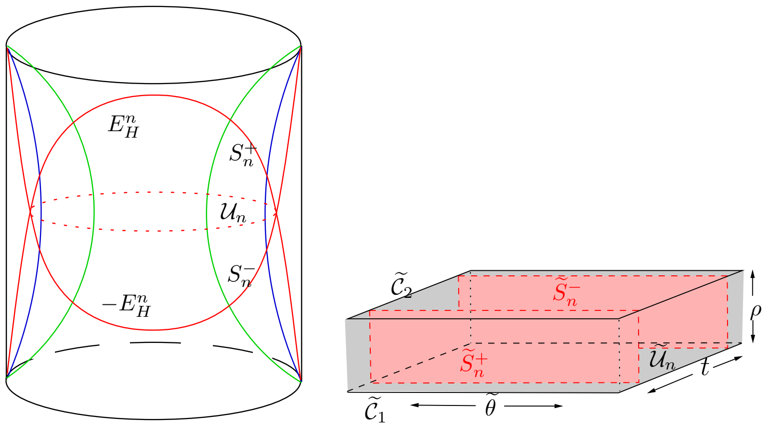

Fig. 3 $\mathcal{U}_n = \Omega \cap \widehat{\mathcal{U}}_n$ and $\widetilde{\mathcal{U}}_n$ denotes its universal cover. Note that $\partial\widetilde{\mathcal{U}}_n \subset \widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2 \cup \widetilde{S}^+_n \cup \widetilde{S}^-_n$

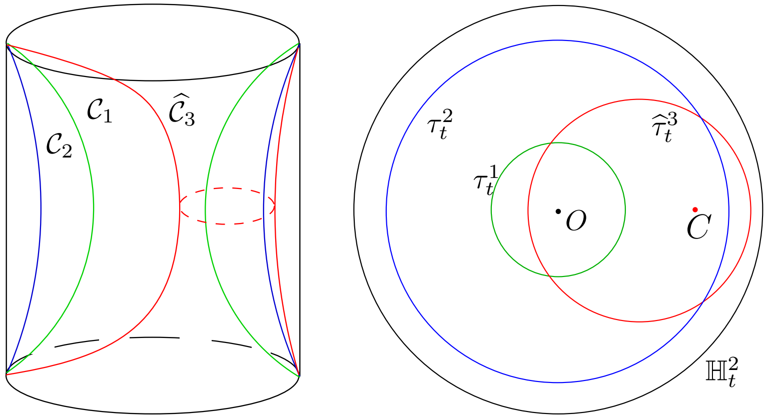

Fig. 4 $\tau_t^i = \mathcal{C}_i \cap \mathbb{H}_t^2$ is a round circle of radius $b_i(t)$ with center O. $\widehat{\tau}_t^3 = \widehat{\mathcal{C}}_3 \cap \mathbb{H}_t^2$ is a round circle of radius $b_2(t)$ with center $C = (\eta_2, 0, t)$

$\widetilde{S}_n^{\pm}$ is an infinite H-strip in $\widetilde{\Omega}$, and the mean curvature vectors of the surfaces $\widetilde{S}_n^{+}$, $\widetilde{S}_n^{-}$ point into $\widetilde{\mathcal{U}}_n$ along $\widetilde{S}_n^{\pm}$. Note that each $\widetilde{\mathcal{U}}_n$ has bounded t-coordinate. Furthermore, we can view $\widetilde{\mathcal{U}}_n$ as $(\mathcal{U}_n \cap \mathcal{P}_0) \times \mathbb{R}$, where $\mathcal{P}_0$ is the half-plane $\{\theta = 0\}$ and the second coordinate is $\widetilde{\theta}$. Abusing the notation, we **redefine** $\widetilde{\mathcal{U}}_n$ to be $\widetilde{\mathcal{U}}_n \cap \widetilde{\Omega}^*$, that is we have removed the infinite bumps $\mathcal{B}^{\pm}$ from $\widetilde{\mathcal{U}}_n$.

Now, we will perform a sequence of modifications of $\widetilde{\mathcal{U}}_n$ so that for each of these modifications, the $\widetilde{\theta}$-coordinate in $\widetilde{\mathcal{U}}_n$ is bounded and so that we obtain a compact exhaustion of $\widetilde{\Omega}^*$. In order to do this, we will use arguments that are similar to those in Claim 3.1. Recall that the necksize of $\mathcal{C}_2$ is $\eta_2 = b_2(0)$. Let $\widehat{\mathcal{C}}_3 = \widehat{\varphi}_{\eta_2}(\mathcal{C}_2)$, see equation (3) for the definition of $\widehat{\varphi}_{\eta_2}$. Then, $\widehat{\mathcal{C}}_3$ is a rotationally invariant catenoid whose rotational axis is the line $(\eta_2, 0) \times \mathbb{R}$ (Fig. 4-left).

Lemma 3.3 *The intersection $\widehat{\mathcal{C}}_3 \cap \Omega$ is a pair of infinite strips.*

Proof It suffices to show that $\widehat{\mathcal{C}}_3 \cap \mathcal{C}_1$ and $\widehat{\mathcal{C}}_3 \cap \mathcal{C}_2$ each consists of a pair of infinite lines. Now, consider the horizontal circles τ_t^1, τ_t^2, and $\widehat{\tau}_t^3$ in the intersection of $\mathbb{H}_t^2$ and $\mathcal{C}_1, \mathcal{C}_2$, and $\widehat{\mathcal{C}}_3$ respectively, where $\mathbb{H}_t^2 = \mathbb{H}^2 \times \{t\}$. For any $t \in \mathbb{R}$, τ_t^i is a circle of radius $b_i(t)$ in $\mathbb{H}_t^2$ with center $(0, 0, t)$. Similarly, $\widehat{\tau}_t^3$ is a circle of radius $b_2(t)$ in $\mathbb{H}_t^2$ with center $(\eta_2, 0, t)$, see Fig. 4-right. Hence, it suffices to show that for any $t \in \mathbb{R}$ each of the intersection $\tau_t^1 \cap \widehat{\tau}_t^3$ and $\tau_t^2 \cap \widehat{\tau}_t^3$ consists of two points.

By construction, it is easy to see $\tau_t^2 \cap \widehat{\tau}_t^3$ consists of two points. This is because τ_t^2 and $\widehat{\tau}_t^3$ have the same radius, $b_2(t)$ and $\eta_2 + b_2(t) > b_2(t)$ and $\eta_2 - b_2(t) > -b_2(t)$. Therefore, it remains to show that $\tau_t^1 \cap \widehat{\tau}_t^3$ consists of two points. By construction, this would be the case if $\eta_2 - b_2(t) < b_1(t)$ and $\eta_2 - b_2(t) > -b_1(t)$. The first inequality follows because $\eta_2 = \inf_{t\in\mathbb{R}} b_2(t)$. The second inequality follows from Lemma 2.1 because

$$\eta_2 > \eta_2 - \eta_1 = \sup_{t\in\mathbb{R}}(b_2(t) - b_1(t)).$$

$\square$

Now, let $\widehat{C}_3 \cap \Omega = T^+ \cup T^-$, where T^+ is the infinite strip with $\theta \in (0, \pi)$, and T^- is the infinite strip with $\theta \in (-\pi, 0)$. Note that $T^\pm$ is a θ-graph over the infinite strip $\widehat{\mathcal{P}}_0 = \Omega \cap \mathcal{P}_0$ where $\mathcal{P}_0$ is the half plane $\{\theta = 0\}$. Let $\mathcal{V}$ be the component of $\Omega - \widehat{C}_3$ containing $\widehat{\mathcal{P}}_0$. Notice that the mean curvature vector $\mathbf{H}$ of $\partial\mathcal{V}$ points into $\mathcal{V}$ on both T^+ and T^-.

Consider the lifts of T^+ and T^- in $\widetilde{\Omega}$. For $n \in \mathbb{Z}$, let $\widetilde{T}_n^+$ be the lift of T^+ which belongs to the region $\widetilde{\theta} \in (2n\pi, (2n+1)\pi)$. Similarly, let $\widetilde{T}_n^-$ be the lift of T^- which belongs to the region $\widetilde{\theta} \in ((2n-1)\pi, 2n\pi)$. Let $\mathcal{V}_n$ be the closed region in $\widetilde{\Omega}$ between the infinite strips $\widetilde{T}_{-n}^-$ and $\widetilde{T}_n^+$. Notice that for n sufficiently large, $\mathcal{B}^\pm \subset \mathcal{V}_n$.

Next we define the compact exhaustion Δ_n of $\widetilde{\Omega}^*$ as follows: $\Delta_n := \widetilde{\mathcal{U}}_n \cap \mathcal{V}_n$. Furthermore, the absolute value of the mean curvature of $\partial\Delta_n$ is equal to H and the mean curvature vector $\mathbf{H}$ of $\partial\Delta_n$ points into Δ_n on $\partial\Delta_n - [(\partial\Delta_n \cap \widetilde{\mathcal{C}}_1) \cup \mathcal{B}^-]$.

3.4 The sequence of H-surfaces

We next define a sequence of compact H-surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ where $\Sigma_n \subset \Delta_n$. For each n sufficiently large, we define a simple closed curve Γ_n in $\partial\Delta_n$, and then we solve the H-Plateau problem for Γ_n in Δ_n. This will provide an embedded H-surface Σ_n in Δ_n with $\partial\Sigma_n = \Gamma_n$ for each n.

The Construction of Γ_n *in* $\partial\Delta_n$:

First, consider the annulus $\mathcal{A}_n = \partial\Delta_n - (\widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2 \cup \mathcal{B}^+ \cup \mathcal{B}^-)$ in $\partial\Delta_n$. Let $\widehat{l}_n^+ = \widetilde{\mathcal{C}}_1 \cap \widetilde{T}_n^+$, and $\widehat{l}_n^- = \widetilde{\mathcal{C}}_2 \cap \widetilde{T}_{-n}^-$ be the pair of infinite lines in $\widetilde{\Omega}$. Let $l_n^\pm = \widehat{l}_n^\pm \cap \mathcal{A}_n$. Let μ_n^+ be an arc in $\widetilde{S}_n^+ \cap \mathcal{A}_n$, whose $\widetilde{\theta}$ and ρ coordinates are strictly increasing as a function of the parameter and whose endpoints are $l_n^+ \cap \widetilde{S}_n^+$ and $l_n^- \cap \widetilde{S}_n^+$ (Fig. 5-left). Similarly, define μ_n^- to be a monotone arc in $\widetilde{S}_n^- \cap \mathcal{A}_n$ whose endpoints are $l_n^+ \cap \widetilde{S}_n^-$ and $l_n^- \cap \widetilde{S}_n^-$. Note that these arcs μ_n^+ and μ_n^- are by construction disjoint from the infinite bumps $\mathcal{B}^\pm$. Then, $\Gamma_n = \mu_n^+ \cup l_n^+ \cup \mu_n^- \cup l_n^-$ is a simple closed curve in $\mathcal{A}_n \subset \partial\Delta_n$ (Fig. 5-right).

Next, consider the following variational problem (H-Plateau problem): Given the simple closed curve Γ_n in $\mathcal{A}_n$, let M be a smooth compact embedded surface in Δ_n with $\partial M = \Gamma_n$. Since Δ_n is simply-connected, M separates Δ_n into two regions. Let Q be the region in $\Delta_n - \Sigma$ with $Q \cap \widetilde{\mathcal{C}}_2 \neq \emptyset$, the "upper" region. Then define the functional $\mathcal{I}_H = \text{Area}(M) + 2H\,\text{Volume}(Q)$.

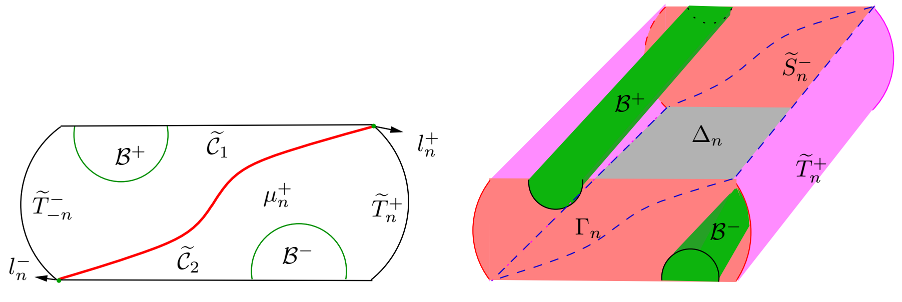

Fig. 5 In the *left*, μ_+^n is pictured in $\widetilde{S}_n^+$. On the *right*, the curve Γ_n is described in $\partial\Delta_n$

By working with integral currents, it is known that there exists a smooth (except at the 4 corners of Γ_n), compact, embedded H-surface $\Sigma_n \subset \Delta_n$ with $\mathrm{Int}(\Sigma_n) \subset \mathrm{Int}(\Delta_n)$ and $\partial \Sigma_n = \Gamma_n$. Note that in our setting, Δ_n is not H-mean convex along $\Delta_n \cap \widetilde{\mathcal{C}}_1$. However, the mean curvature vector along Σ_n points outside Q because of the construction of the variational problem. Therefore $\Delta_n \cap \widetilde{\mathcal{C}}_1$ is still a good barrier for solving the H-Plateau problem. In fact, Σ_n can be chosen to be, and we will assume it is, a minimizer for this variational problem, i.e., $I(\Sigma_n) \leq I(M)$ for any $M \subset \Delta_n$ with $\partial M = \Gamma_n$; see for instance [12, Theorem 2.1] and [1, Theorem 1]. In particular, the fact that $\mathrm{Int}(\Sigma_n) \subset \mathrm{Int}(\Delta_n)$ is proven in Lemma 3 of [4]. Moreover, Σ_n separates Δ_n into two regions.

Similarly to Lemma 4.1 in [3], in the following lemma we show that for any such Γ_n, the minimizer surface Σ_n is a $\widetilde{\theta}$-graph.

Lemma 3.4 *Let $E_n := \mathcal{A}_n \cap \widetilde{T}_n^+$. The minimizer surface Σ_n is a $\widetilde{\theta}$-graph over the compact disk E_n. In particular, the related Jacobi function J_n on Σ_n induced by the inner product of the unit normal field to Σ_n with the Killing field $\partial_{\widetilde{\theta}}$ is positive in the interior of Σ_n.*

Proof The proof is almost identical to the proof of Lemma 4.1 in [3], and for the sake of completeness, we give it here. Let T_α be the isometry of $\widetilde{\Omega}$ which is a translation by α in the $\widetilde{\theta}$ direction, i.e.,

$$T_\alpha(\rho, \widetilde{\theta}, t) = (\rho, \widetilde{\theta} + \alpha, t). \tag{4}$$

Let $T_\alpha(\Sigma_n) = \Sigma_n^\alpha$ and $T_\alpha(\Gamma_n) = \Gamma_n^\alpha$. We claim that $\Sigma_n^\alpha \cap \Sigma_n = \emptyset$ for any $\alpha \in \mathbb{R} \setminus \{0\}$ which implies that Σ_n is a $\widetilde{\theta}$-graph; we will use that Γ_n^α is disjoint from Σ_n for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Arguing by contradiction, suppose that $\Sigma_n^\alpha \cap \Sigma_n \neq \emptyset$ for a certain $\alpha \neq 0$. By compactness of Σ_n, there exists a largest positive number α' such that $\Sigma_n^{\alpha'} \cap \Sigma_n \neq \emptyset$. Let $p \in \Sigma_n^{\alpha'} \cap \Sigma_n$. Since $\partial \Sigma_n^{\alpha'} \cap \partial \Sigma_n = \emptyset$ and the interior of Σ_n, respectively $\Sigma_n^{\alpha'}$, lie in the interior of Δ_n, respectively $T_{\alpha'}(\Delta_n)$, then $p \in \mathrm{Int}(\Sigma_n^{\alpha'}) \cap \mathrm{Int}(\Sigma_n)$. Since the surfaces $\mathrm{Int}(\Sigma_n^{\alpha'})$, $\mathrm{Int}(\Sigma_n)$ lie on one side of each other and intersect tangentially at the point p with the same mean curvature vector, then we obtain a contradiction to the mean curvature comparison principle for constant mean curvature surfaces, see Proposition 2.2. This proves that Σ_n is graphical over its $\widetilde{\theta}$-projection to E_n.

Since by construction every integral curve, $(\overline{\rho}, s, \overline{t})$ with $\overline{\rho}, \overline{t}$ fixed and $(\overline{\rho}, s_0, \overline{t}) \in E_n$ for a certain s_0, of the Killing field $\partial_{\widetilde{\theta}}$ has non-zero intersection number with any compact surface bounded by Γ_n, we conclude that every such integral curve intersects both the disk E_n and Σ_n in single points. This means that Σ_n is a $\widetilde{\theta}$-graph over E_n and thus the related Jacobi function J_n on Σ_n induced by the inner product of the unit normal field to Σ_n with the Killing field $\partial_{\widetilde{\theta}}$ is non-negative in the interior of Σ_n. Since J_n is a non-negative Jacobi function, then either $J_n \equiv 0$ or $J_n > 0$. Since by construction J_n is positive somewhere in the interior, then J_n is positive everywhere in the interior. This finishes the proof of the lemma. $\square$

4 The proof of Theorem 1.1

With Γ_n as previously described, we have so far constructed a sequence of compact stable H-disks Σ_n with $\partial \Sigma_n = \Gamma_n \subset \partial \Delta_n$. Let J_n be the related non-negative Jacobi function described in Lemma 3.4.

By the curvature estimates for stable H-surfaces given in [11], the norms of the second fundamental forms of the Σ_n are uniformly bounded from above at points which are at intrinsic distance at least one from their boundaries. Since the boundaries of the Σ_n leave every compact subset of $\widetilde{\Omega}^*$, for each compact set of $\widetilde{\Omega}^*$, the norms of the second fundamental forms of the Σ_n are uniformly bounded for values n sufficiently large and such a bound does not depend on the chosen compact set. Standard compactness arguments give that, after passing to a subsequence, Σ_n converges to a (weak) H-lamination $\widetilde{\mathcal{L}}$ of $\widetilde{\Omega}^*$ and the leaves of $\widetilde{\mathcal{L}}$ are complete and have uniformly bounded norm of their second fundamental forms, see for instance [5].

Let β be a compact embedded arc contained in $\widetilde{\Omega}^*$ such that its end points p_+ and p_- are contained respectively in $\mathcal{B}^+$ and $\mathcal{B}^-$, and such that these are the only points in the intersection $[\mathcal{B}^+ \cup \mathcal{B}^-] \cap \beta$. Then, for n-sufficiently large, the linking number between Γ_n and β is one, which gives that, for n sufficiently large, Σ_n intersects β in an odd number of points. In particular $\Sigma_n \cap \beta \neq \emptyset$ which implies that the lamination $\widetilde{\mathcal{L}}$ is not empty.

Remark 4.1 By Remark 3.2, a leaf of $\widetilde{\mathcal{L}}$ that is invariant with respect to $\widetilde{\theta}$-translations cannot be contained in $\widetilde{\Omega}^*$. Therefore none of the leaves of $\widetilde{\mathcal{L}}$ are invariant with respect to $\widetilde{\theta}$-translations.

Let $\widetilde{L}$ be a leaf of $\widetilde{\mathcal{L}}$ and let $J_{\widetilde{L}}$ be the Jacobi function induced by taking the inner product of $\partial_{\widetilde{\theta}}$ with the unit normal of $\widetilde{L}$. Then, by the nature of the convergence, $J_{\widetilde{L}} \geq 0$ and therefore since it is a Jacobi field, it is either positive or identically zero. In the latter case, $\widetilde{\mathcal{L}}$ would be invariant with respect to $\widetilde{\theta}$-translations, contradicting Remark 4.1. Thus, by Remark 4.1, we have that $J_{\widetilde{L}}$ is positive and therefore $\widetilde{L}$ is a Killing graph with respect to $\partial_{\widetilde{\theta}}$.

Claim 4.2 *Each leaf $\widetilde{L}$ of $\widetilde{\mathcal{L}}$ is properly embedded in $\widetilde{\Omega}^*$.*

Proof Arguing by contradiction, suppose there exists a leaf $\widetilde{L}$ of $\widetilde{\mathcal{L}}$ that is NOT proper in $\widetilde{\Omega}^*$. Then, since the leaf $\widetilde{L}$ has uniformly bounded norm of its second fundamental form, the closure of $\widetilde{L}$ in $\widetilde{\Omega}^*$ is a lamination of $\widetilde{\Omega}^*$ with a limit leaf Λ, namely $\Lambda \subset \overline{\widetilde{L}} - \widetilde{L}$. Let J_Λ be the Jacobi function induced by taking the inner product of $\partial_{\widetilde{\theta}}$ with the unit normal of Λ.

Just like in the previous discussion, by the nature of the convergence, $J_\Lambda \geq 0$ and therefore, since it is a Jacobi field, it is either positive or identically zero. In the latter case, Λ would be invariant with respect to $\widetilde{\theta}$-translations and thus, by Remark 4.1, Λ cannot be contained in $\widetilde{\Omega}^*$. However, since Λ is contained in the closure of $\widetilde{L}$, this would imply that $\widetilde{L}$ is not contained in $\widetilde{\Omega}^*$, giving a contradiction. Thus, J_Λ must be positive and therefore, Λ is a Killing graph with respect to $\partial_{\widetilde{\theta}}$. However, this implies that $\widetilde{L}$ cannot be a Killing graph with respect to $\partial_{\widetilde{\theta}}$. This follows because if we fix a point p in Λ and let $U_p \subset \Lambda$ be neighborhood of such point, then by the nature of the convergence, U_p is the limit of a sequence of disjoint domains U_{p_n} in $\widetilde{L}$ where $p_n \in \widetilde{L}$ is a sequence of points converging to p and $U_{p_n} \subset \widetilde{L}$ is a neighborhood of p_n. While each domain U_{p_n} is a Killing graph with respect to $\partial_{\widetilde{\theta}}$, the convergence to U_p implies that their union is not. This gives a contradiction and proves that Λ cannot be a Killing graph with respect to $\partial_{\widetilde{\theta}}$. Since we have already shown that Λ must be a Killing graph with respect to $\partial_{\widetilde{\theta}}$, this gives a contradiction. Thus Λ cannot exist and each leaf $\widetilde{L}$ of $\widetilde{\mathcal{L}}$ is properly embedded in $\widetilde{\Omega}^*$. □

Arguing similarly to the proof of the previous claim, it follows that a small perturbation of β, which we still denote by β intersects Σ_n and $\widetilde{\mathcal{L}}$ transversally in a finite number of points. Note that $\widetilde{\mathcal{L}}$ is obtained as the limit of Σ_n. Indeed, since Σ_n separates $\mathcal{B}^+$ and $\mathcal{B}^-$ in $\widetilde{\Omega}^*$, the algebraic intersection number of β and Σ_n must be one, which implies that β intersects Σ_n in an odd number of points. Then β intersects $\widetilde{\mathcal{L}}$ in an odd number of points and the claim below follows.

Claim 4.3 *The curve β intersects $\widetilde{\mathcal{L}}$ in an odd number of points.*

In particular β intersects only a finite collection of leaves in $\widetilde{\mathcal{L}}$ and we let $\mathcal{F}$ denote the non-empty finite collection of leaves that intersect β.

Definition 4.1 Let $(\rho_1, \widetilde{\theta}_0, t_0)$ be a fixed point in $\widetilde{\mathcal{C}}_1$ and let $\rho_2(\widetilde{\theta}_0, t_0) > \rho_1$ such that $(\rho_2(\widetilde{\theta}_0, t_0), \widetilde{\theta}_0, t_0)$ is in $\widetilde{\mathcal{C}}_2$. Then we call the arc in $\widetilde{\Omega}$ given by

$$(\rho_1 + s(\rho_2 - \rho_1), \widetilde{\theta}_0, t_0), \quad s \in [0, 1]. \tag{5}$$

the vertical line segment based at $(\rho_1, \widetilde{\theta}_0, t_0)$.

Claim 4.4 *There exists at least one leaf $\widetilde{L}_\beta$ in $\mathcal{F}$ that intersects β in an odd number of points and the leaf $\widetilde{L}_\beta$ must intersect each vertical line segment at least once.*

Proof The existence of $\widetilde{L}_\beta$ follows because otherwise, if all the leaves in $\mathcal{F}$ intersected β in an even number of points, then the number of points in the intersection $\beta \cap \mathcal{F}$ would be even. Given $\widetilde{L}_\beta$ a leaf in $\mathcal{F}$ that intersects β in an odd number of points, suppose there exists a vertical line segment which does not intersect $\widetilde{L}_\beta$. Then since by Claim 4.2 $\widetilde{L}_\beta$ is properly embedded, using elementary separation arguments would give that the number of points of intersection in $\beta \cap \widetilde{L}_\beta$ must be zero mod 2, that is even, contradicting the previous statement. □

Let Π be the covering map defined in equation (2) and let $\mathcal{P}_H := \Pi(\widetilde{L}_\beta)$. The previous discussion and the fact that Π is a local diffeomorphism, implies that $\mathcal{P}_H$ is a stable complete H-surface embedded in Ω. Indeed, $\mathcal{P}_H$ is a graph over its θ-projection to $\mathrm{Int}(\Omega) \cap \{(\rho, 0, t) \mid \rho > 0, t \in \mathbb{R}\}$, which we denote by $\theta(\mathcal{P}_H)$. Abusing the notation, let $J_{\mathcal{P}_H}$ be the Jacobi function induced by taking the inner product of ∂_θ with the unit normal of $\mathcal{P}_H$, then $J_{\mathcal{P}_H}$ is positive. Finally, since the norm of the second fundamental form of $\mathcal{P}_H$ is uniformly bounded, standard compactness arguments imply that its closure $\overline{\mathcal{P}}_H$ is an H-lamination $\mathcal{L}$ of Ω, see for instance [5].

Claim 4.5 *The closure of $\mathcal{P}_H$ is an H-lamination of Ω consisting of itself and two H-catenoids $L_1, L_2 \subset \Omega$ that form the limit set of $\mathcal{P}_H$.*

Remark 4.6 Note that these two H-catenoids are not necessarily the ones which determine $\partial\Omega$.

Proof Given $(\rho_1, \widetilde{\theta}_0, t_0) \in \widetilde{\mathcal{C}}_1$, let $\widetilde{\gamma}$ be the fixed vertical line segment in $\widetilde{\Omega}$ based at $(\rho_1, \widetilde{\theta}_0, t_0)$, let $\widetilde{p}_0$ be a point in the intersection $\widetilde{L}_\beta \cap \widetilde{\gamma}$ (recall that by Claim 4.4 such intersection is not empty) and let $p_0 = \Pi(\widetilde{p}_0) \in \Pi(\widetilde{\gamma}) \cap \mathcal{P}_H$. Then, by Claim 4.4, for any $i \in \mathbb{N}$, the vertical line segment $T_{2\pi i}(\widetilde{\gamma})$ intersects $\widetilde{L}_\beta$ in at least a point $\widetilde{p}_i$, and $\widetilde{p}_{i+1}$ is above $\widetilde{p}_i$, where T is the translation defined in equation (4). Namely, $\widetilde{p}_0 = (r_0, \widetilde{\theta}_0, t_0)$, $\widetilde{p}_i = (r_i, \widetilde{\theta}_0 + 2\pi i, t_0)$ and $r_i < r_{i+1} < \rho_2(\widetilde{\theta}_0, t_0)$. The point $\widetilde{p}_i \in \widetilde{L}_\beta$ corresponds to the point $p_i = \Pi(\widetilde{p}_i) = (r_i, \widetilde{\theta}_0 \bmod 2\pi, t_0) \in \mathcal{P}_H$. Let $r(2) := \lim_{i\to\infty} r_i$ then $r(2) \leq \rho_2(\widetilde{\theta}_0, t_0)$ and note that since $\lim_{i\to\infty}(r_{i+1} - r_i) = 0$, then the value of the Jacobi function $J_{\mathcal{P}_H}$ at p_i must be going to zero as i goes to infinity. Clearly, the point $Q := (r(2), \widetilde{\theta}_0 \bmod 2\pi, t_0) \in \Omega$ is in the closure of $\mathcal{P}_H$, that is $\mathcal{L}$. Let L_2 be the leaf of $\mathcal{L}$ containing Q. By the previous discussion $J_{L_2}(Q) = 0$. Since by the nature of the convergence, either J_{L_2} is positive or L_2 is rotational, then L_2 is rotational, namely an H-catenoid.

Arguing similarly but considering the intersection of $\widetilde{L}_\beta$ with the vertical line segments $T_{-2\pi i}(\widetilde{\gamma})$, $i \in \mathbb{N}$, one obtains another H-catenoid L_1, different from L_2, in the lamination $\mathcal{L}$. This shows that the closure of $\mathcal{P}_H$ contains the two H-catenoids L_1 and L_2.

Let Ω_g be the rotationally invariant, connected region of $\Omega - [L_1 \cup L_2]$ whose boundary contains $L_1 \cup L_2$. Note that since $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ is contained in its closure, then $\mathcal{P}_H \subset \Omega_g$. It remains to show that $\mathcal{L} = \mathcal{P}_H \cup L_1 \cup L_2$, i.e. $\overline{\mathcal{P}}_H - \mathcal{P}_H = L_1 \cup L_2$. If $\overline{\mathcal{P}}_H - \mathcal{P}_H \neq L_1 \cup L_2$ then there would be another leaf $L_3 \in \mathcal{L} \cap \Omega_g$ and by previous argument, L_3 would be an H-catenoid. Thus L_3 would separate Ω_g into two regions, contradicting that fact that $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ are contained in its closure. This finishes the proof of the claim. □

Note that by the previous claim, $\mathcal{P}_H$ is properly embedded in Ω_g.

Claim 4.7 *The H-surface $\mathcal{P}_H$ is simply-connected and every integral curve of ∂_θ that lies in Ω_g intersects $\mathcal{P}_H$ in exactly one point.*

Proof Let $D_g := \mathrm{Int}(\Omega_g) \cap \{(\rho, 0, t) \mid \rho > 0,\ t \in \mathbb{R}\}$, then $\mathcal{P}_H$ is a graph over its θ-projection to D_g, that is $\theta(\mathcal{P}_H)$. Since $\theta\colon \Omega_g \to D_g$ is a proper submersion and $\mathcal{P}_H$ is properly embedded in Ω_g, then $\theta(\mathcal{P}_H) = D_g$, which implies that every integral curve of ∂_θ that lies in Ω_g intersects $\mathcal{P}_H$ in exactly one point. Moreover, since D_g is simply-connected, this gives that $\mathcal{P}_H$ is also simply-connected. This finishes the proof of the claim. □

From this claim, it clearly follows that Ω_g is foliated by H-surfaces, where the leaves of this foliation are L_1, L_2 and the rotated images $\mathcal{P}_H(\theta)$ of $\mathcal{P}_H$ around the t-axis by angles $\theta \in [0, 2\pi)$. The existence of the examples Σ_H in the statement of Theorem 1.1 can easily be proven by using $\mathcal{P}_H$. We set $\Sigma_H = \mathcal{P}_H$, and $C_i = L_i$ for $i = 1, 2$. This finishes the proof of Theorem 1.1.

Appendix: Disjoint H-catenoids

In this section, we will show the existence of disjoint H-catenoids in $\mathbb{H}^2 \times \mathbb{R}$. In particular, we will prove Lemma 2.1. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, recall that $\eta_d = \cosh^{-1}(\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2})$ and that $\lambda_d \colon [\eta_d, \infty) \to [0, \infty)$ is the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr. \tag{6}$$

Recall that $\lambda_d(\rho)$ is a monotone increasing function with $\lim_{\rho\to\infty} \lambda_d(\rho) = \infty$ and that $\lambda_d'(\eta_d) = \infty$ when $d \in (-2H, \infty)$. The H-catenoid $\mathcal{C}_d^H$, $d \in (-2H, \infty)$, is obtained by rotating a generating curve $\widehat{\lambda}_d(\rho)$ about the t-axis. The generating curve $\widehat{\lambda}_d$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$.

Finally, recall that $b_d(t) := \lambda_d^{-1}(t)$ for $t \geq 0$, hence $b_d(0) = \eta_d$, and that abusing the notation $b_d(t) := b_d(-t)$ for $t \leq 0$.

Lemma 2.1 (Disjoint H-catenoids) Given $d_1 > 2$ there exist $d_0 > d_1$ and $\delta_0 > 0$ such that for any $d_2 \in [d_0, \infty)$ and $t > 0$ then

$$\inf_{t\in\mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.$$

In particular, the corresponding H-catenoids are disjoint, i.e., $\mathcal{C}_{d_1}^H \cap \mathcal{C}_{d_2}^H = \emptyset$.

Moreover, $b_{d_2}(t) - b_{d_1}(t)$ is decreasing for $t > 0$ and increasing for $t < 0$. In particular,

$$\sup_{t\in\mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.$$

Proof We begin by introducing the following notations that will be used for the computations in the proof of this lemma,

$$c := \cosh r = \frac{e^r + e^{-r}}{2}, \quad s := \sinh r = \frac{e^r - e^{-r}}{2}.$$

Recall that $c^2 - s^2 = 1$ and $c - s = e^{-r}$. Using these notations,

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} \, dr \tag{7}$$

can be rewritten as

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H(s + e^{-r})}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr = f_d(\rho) + J_d(\rho), \tag{8}$$

where

$$f_d(\rho) = \int_{\eta_d}^{\rho} \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}}\, dr \quad \text{and} \quad J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{s^2 - (d + 2Hc)^2}}\, dr$$

First, by using a series of substitutions, we will get an explicit description of $f_d(\rho)$. Then, we will show that for $d > 2$, $J_d(\rho)$ is bounded independently of ρ and d.

Claim 4.8

$$f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1}\left(\frac{(1 - 4H^2)\cosh\rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right). \tag{9}$$

Remark 4.9 After finding $f_d(\rho)$, we used Wolfram Alpha to compute the derivative of $f_d(\rho)$ and verify our claim. For the sake of completeness, we give a proof.

Proof of Claim 4.8 The proof is a computation with requires several integrations by substitution. Consider

$$\int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}}\, dr$$

By using the fact that $s^2 = c^2 - 1$ and applying the substitution $\{u = c, du = \frac{dc}{dr} dr = s dr\}$ we obtain that

$$\int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}}\, dr = \int \frac{2H}{\sqrt{u^2 - 1 - (d + 2Hu)^2}}\, du.$$

Note that

$$\begin{aligned}
&u^2 - 1 - (d + 2Hu)^2 = u^2 - 1 - (d^2 + 4dHu + 4H^2u^2) \\
&= (1 - 4H^2)u^2 - 4dHu - d^2 - 1 \\
&= (1 - 4H^2)\left(u^2 - \frac{4dH}{1 - 4H^2}u + \frac{4d^2H^2}{(1 - 4H^2)^2}\right) - \frac{4d^2H^2}{1 - 4H^2} - d^2 - 1 \\
&= (1 - 4H^2)\left[\left(u - \frac{2dH}{(1 - 4H^2)}\right)^2 - \left(\frac{4d^2H^2}{(1 - 4H^2)^2} + \frac{d^2 + 1}{1 - 4H^2}\right)\right] \\
&= (1 - 4H^2)\left[\left(u - \frac{2dH}{(1 - 4H^2)}\right)^2 - \left(\frac{4d^2H^2 + (1 - 4H^2)(d^2 + 1)}{(1 - 4H^2)^2}\right)\right] \\
&= (1 - 4H^2)\left[\left(u - \frac{2dH}{(1 - 4H^2)}\right)^2 - \left(\frac{d^2 + 1 - 4H^2}{(1 - 4H^2)^2}\right)\right].
\end{aligned}$$

Therefore, by applying a second substitution, $\{w = u - \frac{2dH}{(1-4H^2)}, dw = du\}$, and letting $a^2 = (\frac{d^2+1-4H^2}{(1-4H^2)^2})$ we get that

$$\int \frac{2H}{\sqrt{u^2-1-(d+2Hu)^2}}\, du = \int \frac{2H}{\sqrt{1-4H^2}\sqrt{w^2-a^2}}\, dw$$

By using the fact that $\sec^2 x - 1 = \tan^2 x$ and applying a third substitution, $\{w = a \sec t, dw = a \sec t \tan t dt\}$, we obtain that

$$\begin{aligned}\int \frac{2Ha \sec t \tan t}{\sqrt{1-4H^2}\sqrt{a^2 \sec^2 t - a^2}} dt &= \int \frac{2H \sec t}{\sqrt{1-4H^2}}\, dt \\ &= \frac{2H}{\sqrt{1-4H^2}} \ln|\sec t + \tan t|\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{2H}{\sqrt{1-4H^2}\sqrt{w^2-a^2}}\, dw &= \frac{2H}{\sqrt{1-4H^2}} \ln|\frac{w}{a} + \sqrt{\frac{w^2}{a^2}-1}| \\ &= \frac{2H}{\sqrt{1-4H^2}} \cosh^{-1}\left(\frac{w}{a}\right)\end{aligned}$$

Since $w = u - \frac{2dH}{(1-4H^2)}$ then

$$\begin{aligned}\int \frac{2H}{\sqrt{u^2-1-(d+2Hu)^2}}\, du &= \frac{2H}{\sqrt{1-4H^2}} \cosh^{-1}\left(\frac{u - \frac{2dH}{(1-4H^2)}}{a}\right) \\ &= \frac{2H}{\sqrt{1-4H^2}} \cosh^{-1}\left(\frac{u - \frac{2dH}{(1-4H^2)}}{\frac{\sqrt{d^2+1-4H^2}}{(1-4H^2)}}\right) \\ &= \frac{2H}{\sqrt{1-4H^2}} \cosh^{-1}\left(\frac{(1-4H^2)u - 2dH}{\sqrt{d^2+1-4H^2}}\right).\end{aligned}$$

Finally, since $u = \cosh r$

$$\begin{aligned}\int_{\eta_d}^{\rho} \frac{2Hs}{\sqrt{s^2-(d+2Hc)^2}} &= \frac{2H}{\sqrt{1-4H^2}} \cosh^{-1}\left(\frac{(1-4H^2)\cosh r - 2dH}{\sqrt{d^2+1-4H^2}}\right)\Bigg|_{\eta_d}^{\rho} \\ &= \frac{2H}{\sqrt{1-4H^2}} \left(\cosh^{-1}\left(\frac{(1-4H^2)\cosh \rho - 2dH}{\sqrt{d^2+1-4H^2}}\right)\right. \\ &\quad \left. - \cosh^{-1}\left(\frac{(1-4H^2)\cosh \eta_d - 2dH}{\sqrt{d^2+1-4H^2}}\right)\right)\end{aligned}$$

Recall that $\eta_d = \cosh^{-1}(\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2})$ and thus

$$\frac{(1-4H^2)\cosh\eta_d - 2dH}{\sqrt{d^2+1-4H^2}} = \frac{(1-4H^2)(\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}) - 2dH}{\sqrt{d^2+1-4H^2}} = 1.$$

This implies that

$$f_d(\rho) = \frac{2H}{\sqrt{1-4H^2}}\cosh^{-1}\left(\frac{(1-4H^2)\cosh\rho - 2dH}{\sqrt{d^2+1-4H^2}}\right).$$

□

By Claim 4.8 we have that

$$\begin{aligned} f_d(\rho) &= \frac{2H}{\sqrt{1-4H^2}}\left(\cosh^{-1}\frac{(1-4H^2)\cosh\rho - 2dH}{\sqrt{d^2+1-4H^2}}\right) \\ &= \frac{2H}{\sqrt{1-4H^2}}\left(\rho + \ln\frac{1-4H^2}{\sqrt{d^2+1-4H^2}}\right) + g_d(\rho), \end{aligned}$$

where $\lim_{\rho\to\infty} g_d(\rho) = 0$.

Recall that $\lambda_d(\rho) = f_d(\rho) + J_d(\rho)$ where

$$J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d+2He^{-r}}{\sqrt{s^2-(d+2Hc)^2}}\,dr = \int_{\eta_d}^{\rho} \frac{d+2He^{-r}}{\sqrt{c^2-1-(d+2Hc)^2}}\,dr.$$

Claim 4.10

$$\sup_{d\in(2,\infty),\rho\in(\eta_d,\infty)} J_d(\rho) \leq \pi\sqrt{1-2H}.$$

Proof of Claim 4.10 Let

$$\alpha = \frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} \text{ and } \beta = \frac{2dH-\sqrt{1-4H^2+d^2}}{1-4H^2}$$

be the roots of $c^2-1-(d+2Hc)^2$, i.e.

$$\begin{aligned} c^2-1-(d+2Hc)^2 &= (1-4H^2)\left(c^2 - \frac{4dH}{1-4H^2}c - \frac{1+d^2}{1-4H^2}\right) \\ &= (1-4H^2)(c-\alpha)(c-\beta). \end{aligned}$$

Note that $\alpha = \cosh \eta_d$ and that as $H \in (0, \frac{1}{2})$, $\beta < 0 < \alpha$. Furthermore, $2He^{-r} < 2H < 1 < d$. Thus we have,

$$\begin{aligned} J_d(\rho) &= \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{1-4H^2}\sqrt{(c-\alpha)(c-\beta)}} dr \\ &< \frac{2d}{\sqrt{1-4H^2}} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{(c-\alpha)(c-\beta)}} \\ &< \frac{2d}{\sqrt{1-4H^2}\sqrt{\alpha-\beta}} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c-\alpha}}, \end{aligned}$$

where the last inequality holds because for $r > \eta_d$, $\cosh r > \alpha$ and thus $\sqrt{\alpha-\beta} < \sqrt{c-\alpha}$. Notice that $\alpha - \beta = \frac{2\sqrt{1-4H^2+d^2}}{1-4H^2} > \frac{2d}{1-4H^2}$. Therefore

$$\frac{2d}{\sqrt{1-4H^2}\sqrt{\alpha-\beta}} < \frac{2d}{\sqrt{1-4H^2}} \frac{\sqrt{1-4H^2}}{\sqrt{2d}} = \sqrt{2d}$$

and

$$J_d(\rho) < \sqrt{2d} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c-\alpha}}.$$

Applying the substitution $\{u = c - \alpha, du = sdr = \sqrt{(u+\alpha)^2 - 1}dr\}$, we obtain that

$$\int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c-\alpha}} = \int_0^{\infty} \frac{du}{\sqrt{u}\sqrt{(u+\alpha)^2-1}} \tag{10}$$

Let $\omega = \alpha - 1$. Note that since $d \geq 1$ then $\alpha > 1$ and we have that $(u+\alpha)^2 - 1 > (u+\omega)^2$ as $u > 0$. This gives that

$$\int_0^{\infty} \frac{du}{\sqrt{u}\sqrt{(u+\alpha)^2-1}} < \int_0^{\infty} \frac{du}{\sqrt{u}(u+\omega)}$$

Applying the substitution $\{v = \sqrt{u}, dv = \dfrac{du}{2\sqrt{u}}\}$ we get

$$\int_0^{\infty} \frac{du}{\sqrt{u}(u+\omega)} = \int_0^{\infty} \frac{2dv}{v^2+\omega} = \frac{2}{\sqrt{\omega}} \arctan \frac{w}{\sqrt{\omega}}\bigg|_0^{\infty} < \frac{\pi}{\sqrt{\omega}}$$

and thus

$$J_d(\rho) < \sqrt{\frac{2d}{\omega}}\pi.$$

Note that

$$\omega = \alpha - 1 = \frac{2dH + \sqrt{1-4H^2+d^2}}{1-4H^2} - 1$$
$$> \frac{(1+2H)d}{1-4H^2} - 1 = \frac{d}{1-2H} - 1.$$

Since $d > 2$, we have $2\omega > \dfrac{d}{1-2H}$ and $\dfrac{d}{\omega} < 2(1-2H)$. Then $\sqrt{\dfrac{2d}{\omega}} < 2\sqrt{1-2H}$.

Finally, this gives that

$$J_d(\rho) < 2\pi\sqrt{1-2H}$$

independently on $d > 2$ and $\rho > \eta_d$. This finishes the proof of the claim. □

Using Claims 4.8 and 4.10, we can now prove the next claim.

Claim 4.11 *Given $d_2 > d_1 > 2$ there exists $T \in \mathbb{R}$ such for any $t > T$, we have that*

$$\frac{2H}{\sqrt{1-4H^2}}(\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t))$$
$$> \frac{1}{2}\ln\sqrt{\frac{d_2^2+1-4H^2}{d_1^2+1-4H^2}} - 2\pi\sqrt{1-2H}.$$

Proof of Claim 4.11 Recall that $\lambda_d(\rho) = f_d(\rho) + J_d(\rho)$ and that by Claims 4.8 and 4.10 we have that

$$f_d(\rho) = \frac{2H}{\sqrt{1-4H^2}}\left(\rho + \ln\frac{1-4H^2}{\sqrt{d^2+1-4H^2}}\right) + g_d(\rho), \tag{11}$$

where $\lim_{\rho\to\infty} g_d(\rho) = 0$, and that

$$\sup_{d\in(2,\infty),\rho\in(\eta_d,\infty)} J_d(\rho) \leq 2\pi\sqrt{1-2H}. \tag{12}$$

Let $\rho_i(t) := \lambda_{d_i}^{-1}(t)$, $i = 1, 2$. Using this notation, since $t = \lambda_1(\rho_1(t)) = \lambda_2(\rho_2(t))$ we obtain that

$$\begin{aligned}
0 &= \lambda_2(\rho_2(t)) - \lambda_1(\rho_1(t))\\
&= f_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t)) - f_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t))\\
&= \frac{2H}{\sqrt{1-4H^2}}\left(\rho_2(t) + \ln\frac{1-4H^2}{\sqrt{d_2^2+1-4H^2}}\right) + g_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t))\\
&\quad - \frac{2H}{\sqrt{1-4H^2}}\left(\rho_1(t) - \ln\frac{1-4H^2}{\sqrt{d_1^2+1-4H^2}}\right) - g_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t))
\end{aligned}$$

Recall that $\lim_{t\to\infty}\rho_i(t)=\infty$, $i=1,2$, therefore given $\varepsilon>0$ there exists $T_\varepsilon\in\mathbb{R}$ such that for any $t>T_\varepsilon$, $|g_{d_i}(\rho_i(t))|\le\varepsilon$. Taking

$$4\varepsilon<\ln\sqrt{\frac{d_2^2+1-4H^2}{d_1^2+1-4H^2}}$$

for $t>T_\varepsilon$ we get that

$$\begin{aligned}\frac{2H}{\sqrt{1-4H^2}}&(\rho_2(t)-\rho_1(t))\\&>\ln\sqrt{\frac{d_2^2+1-4H^2}{d_1^2+1-4H^2}}+J_{d_1}(\rho_1(t))-J_{d_2}(\rho_2(t))-2\varepsilon\\&>\frac{1}{2}\ln\sqrt{\frac{d_2^2+1-4H^2}{d_1^2+1-4H^2}}+J_{d_1}(\rho_1(t))-J_{d_2}(\rho_2(t)).\end{aligned}$$

Notice that $J_{d_1}(\rho_1(t))>0$ and that Claim 4.10 gives that

$$\sup_{\rho\in(\eta_{d_2},\infty)}J_{d_2}(\rho)\le 2\pi\sqrt{1-2H}.$$

Therefore

$$\begin{aligned}\frac{2H}{\sqrt{1-4H^2}}&(\rho_2(t)-\rho_1(t))\\&>\frac{1}{2}\ln\sqrt{\frac{d_2^2+1-4H^2}{d_1^2+1-4H^2}}-2\pi\sqrt{1-2H}.\end{aligned}$$

This finishes the proof of the claim. □

We can now use Claim 4.11 to finish the proof of the lemma. Given $d_1>2$ fix $d_0>d_1$ such that

$$\frac{\sqrt{1-4H^2}}{4H}\left(\ln\sqrt{\frac{d_0^2+1-4H^2}{d_1^2+1-4H^2}}-4\pi\sqrt{1-2H}\right)=1.$$

Then, by Claim 4.11, given $d_2\ge d_0$ there exists $T>0$ such that $\lambda_{d_2}^{-1}(t)-\lambda_{d_1}^{-1}(t)>1$ for any $t>T$. Notice that since for any $\rho\in(\eta_2,\infty)$, $\lambda'_{d_2}(\rho)>\lambda'_{d_1}(\rho)$, then there exists at most one $t_0>0$ such that $\lambda_{d_2}^{-1}(t_0)-\lambda_{d_1}^{-1}(t_0)=0$. Therefore, since there exists $T>0$ such that $\lambda_{d_2}^{-1}(t)-\lambda_{d_1}^{-1}(t)>1$ for any $t>T$ and $\lambda_{d_2}^{-1}(0)-\lambda_{d_1}^{-1}(0)=\eta_{d_2}-\eta_{d_1}>0$, this implies that there exists a constant $\delta(d_2)>0$ such that for any $t>0$,

$$\lambda_{d_2}^{-1}(t)-\lambda_{d_1}^{-1}(t)>\delta(d_2).$$

A priori it could happen that $\lim_{d_2 \to \infty} \delta(d_2) = 0$. The fact that $\lim_{d_2 \to \infty} \delta(d_2) > 0$ follows easy by noticing that by applying Claim 4.11 and using the same arguments as in the previous paragraph there exists $d_3 > d_0$ such that for any $d \geq d_3$ and $t > 0$,

$$\lambda_d^{-1}(t) - \lambda_{d_0}^{-1}(t) > 0.$$

Therefore, for any $d \geq d_3$ and $t > 0$,

$$\lambda_d^{-1}(t) - \lambda_{d_1}^{-1}(t) > \lambda_{d_0}^{-1}(t) - \lambda_{d_1}^{-1}(t) > \delta(d_0)$$

which implies that

$$\lim_{d_2 \to \infty} \delta(d_2) \geq \delta(d_0) > 0.$$

Setting $\delta_0 = \inf_{d \in [d_0, \infty)} \delta(d_2) > 0$ gives that

$$\inf_{t \in \mathbb{R}_{\geq 0}} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.$$

By definition of $b_d(t)$ then

$$\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = \inf_{t \in \mathbb{R}_{\geq 0}} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.$$

It remains to prove that $b_2(t) - b_1(t)$ is decreasing for $t > 0$ and increasing for $t < 0$. By definition of $b_d(t)$, it suffices to show that $b_2(t) - b_1(t)$ is decreasing for $t > 0$. We are going to show $\frac{d}{dt}(b_2(t) - b_1(t)) < 0$ when $t > 0$.

By definition of b_i, for $t > 0$ we have that $\lambda_i(b_i(t)) = t$ and thus $b_i'(t) = \frac{1}{\lambda_i'(b_i(t))}$. By definition of $\lambda_d(t)$ for $t > 0$ the following holds,

$$b_1'(t) = \frac{1}{\lambda_1'(b_1(t))} > \frac{1}{\lambda_1'(b_2(t))} > \frac{1}{\lambda_2'(b_2(t))} = b_2'(t).$$

The first inequality is due to the convexity of the function $\lambda_1(t)$ and the second inequality is due to the fact that $\lambda_1'(\rho) < \lambda_2'(\rho)$ for any $\rho > \eta_2$. This proves that $\frac{d}{dt}(b_2(t) - b_1(t)) = b_2'(t) - b_1'(t) < 0$ for $t > 0$ and finishes the proof of the claim. $\square$

Note that if d is sufficiently close to $-2H$ then $\mathcal{C}_d^H$ must be unstable. This follows because as d approaches $-2H$, the norm of the second fundamental form of $\mathcal{C}_d^H$ becomes arbitrarily large at points that approach the "origin" of $\mathbb{H}^2 \times \mathbb{R}$ and a simple rescaling argument gives that a sequence of subdomains of $\mathcal{C}_d^H$ converge to a catenoid, which is an unstable minimal surface. This observation, together with our previous lemma suggests the following conjecture.

Conjecture: Given $H \in (0, \frac{1}{2})$ there exists $d_H > -2H$ such that the following holds. For any $d > d' > d_H$, $\mathcal{C}_d^H \cap \mathcal{C}_{d'}^H = \emptyset$, and the family $\{\mathcal{C}_d^H \mid d \in [d_H, \infty)\}$ gives a foliation of the closure of the non-simply-connected component of $\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}^H_{d_H}$. The H-catenoid $\mathcal{C}^H_d$ is unstable if $d \in (-2H, d_H)$ and stable if $d \in (d_H, \infty)$. The H-catenoid $\mathcal{C}^H_{d_H}$ is a stable-unstable catenoid.

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